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The Common Nature of Space and Time

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Abstract: *The term quadrivium refers to the study of arithmetic, music, geometry, and astronomy. The Pythagoreans referred to the study of number which are Arithmetic (Number in itself), Music (Number in time), Geometry (Number in space) and Astronomy (Number in space and time). This paper presents the quadrivium as the foundation of a multidisciplinary approach to teaching basic mathematics. Through a brief but in-depth examination of two phenomena, the √2 geometric progression (a spatial sequence), and the musical overtone series (a temporal sequence) are presented in this paper. This paper presents a new relationship between number, geometry and music. While each is rather commonly known within the context of their separate disciplines, it is when combined that a remarkable relationship is uncovered.*

Keywords: Quadrivium; Pythagoras; Boethius; overtone series; √2 geometric sequence; harmonics; orbital resonance; temporal; spatial

Introduction

The relationship between music and mathematics is an ancient topic of study. The Roman diplomat and scholar, Anicius Manlius Severinus Boethius (480–524 CE), wrote extensively on the subject. He coined the term *Quadrivium*, referring to the disciplines of arithmetic, music, geometry and astronomy. Boethius further considered the quadrivium as the fourfold way to real knowledge (Grant, 1999, 102). Centuries later the quadrivium constituted four of the original seven liberal arts (Grant, 1999, 103).

Prior to Boethius, the Greek philosopher and mathematician Pythagoras (ca. 569–475 BCE) also expounded upon these subjects. Although he is not known to have written any books (Guthrie, 1987, p. 19), his followers, the Pythagoreans, left behind a body of writings on the subject. They are said to having viewed these four subjects as follows: (1) Arithmetic – Number in itself, (2) Music – Number in time, (3) Geometry -- Number in space and (4) Astronomy -- Number in space and time (Guthrie, 1987, 34).

This paper offers a contemporary, introductory examination of the four disciplines combined into multidisciplinary outcomes from the areas of arithmetic, music, geometry, and astronomy which further illustrates their relevance and importance when studied *together* in relation to one another, rather than as separate disciplines. This paper is not a comprehensive examination of the quadrivium or the relationship between arithmetic, music, geometry, and astronomy. This multi-disciplinary examination reveals relationships among these disciplines that are perhaps new to our modern body of knowledge. Though evidently well known in the distant past, this study yields a result that greatly exceeds any result obtainable by studying these disciplines separately. The conclusions are interesting and have far-reaching implications.

Thus, this paper focuses on two primary ideas (1) the musical *overtone series* and (2) the *√2 geometric sequence*. The overtone series is fundamental to music and commonly known amongst musicians, physicists, mathematicians, architects, engineers, etc. The √2 geometric sequence is a fundamental and commonly known geometric pattern. Neither is rare nor unique; however, in combination, they produce a result that is both simple and complex with wide-ranging implications.

The primary ideas expressed in this paper are based on pure and simple mathematics. However, the mathematical relationships have implications that transcend the limitations of the individual disciplines that constitute the quadrivium.

The Overtone Series

Most studies of Western music begin by learning or being taught the scale, specifically, the major scale. However, a more comprehensive understanding of the music of any culture would be gained by beginning with the *overtone series*.

The overtone series precedes scales and temperament. Scales are the result of tempering, of some form of temperament or recalibration. Western music is based on equal temperament, a system of tuning that divides our musical octave into twelve equal half steps. The major scale and the chromatic scale are both the result of this system of tempering. To the contrary, the overtone series is an untempered, natural occurrence.

Most trained musicians are familiar with the overtone series. Their knowledge, however, being of a more practical nature, has mainly to do with music performance. Consequently, it is uncommon for the average musician to have anything but a very basic practical knowledge of the overtone series and unusual for one to understand its mechanics. It is extremely rare to find the overtone series being taught as a regular part of any university's curriculum. Musicians will frequently hear overtones and consider them during a musical performance, but an actual study of the overtone series is rarely required and seldom pursued.

Whenever a musical instrument or human voice produces a tone it is not a single, pure tone. Within that tone is a potentially infinite number of higher tones called *overtones*. This sequence of higher tones issue from the original or fundamental tone in a specific ascending order (Martineau, Lundy, Sutton and Ashton, 2011, 247; Botts, 1974, 75-76). These tones are clearly audible when listening to an example of Tuvan Throat Singing or other forms of harmonic chant such as those by David Hykes and the Harmonic Choir. Through the study of certain vocal techniques, various cultures have developed styles of singing that accentuate these overtones.

The best way to illustrate what overtones are and how they are produced is to use the example of a vibrating string. For the sake of example, a hypothetical string will be four feet long and when plucked will vibrate at 100 Hz (vibrations per second) and produce the note C. This particular note C hence becomes the *fundamental* pitch of this overtone series, i.e., all subsequent pitches in the entire series will be based on this fundamental pitch C. The note C does not actually vibrate at 100 Hz. This frequency is being used only to simplify the math used in the examples and illustrations.

Each overtone vibrates at a rate that is an exact multiple of the fundamental note C (Botts, 1974). Therefore, since the fundamental pitch C vibrates at 100 Hz, all subsequent overtones will vibrate at a rate that is a multiple of 100 Hz, as shown in Figure 1.

1. C (100 Hz) (Fundamental pitch)
2. C^1 (200 Hz)
3. G (300 Hz)
4. C^2 (400 Hz)
5. E (500 Hz)
6. G^1 (600 Hz)
7. $B^{\mathbf{b}}$ (700 Hz)
8. C^3 (800 Hz)
9. Etc.....

Figure 1. First eight tones in an overtone series

Figure 1 shows the first eight tones in an overtone series based on the note C. Since the fundamental pitch C vibrates at 100 hertz, the second note in the series will vibrate at 200 Hz, that is, at twice the frequency of the fundamental pitch. The third note in this series, the note G, vibrates at 300 Hz, three times faster than the fundamental pitch C. The fourth note vibrates at 400 Hz, four times faster than the fundamental pitch. The 16^{th} pitch in the series would vibrate at 1,600 hertz, i.e., 16 times faster than the fundamental pitch C (Botts, 1974, pp. 75-76). Therefore, the overtone is an arithmetic sequence where the difference between each successive pitch in the sequence will be the rate of vibration of the fundamental pitch. In this case that difference is 100Hz (Pierce, 2018a).

Since the fundamental pitch, the note C vibrates at 100 Hz, this frequency becomes the basic unit of measure concerning the frequency of all subsequent pitches contained within this overtone series. Figure 2 is simply a reformulation of Figure 1, presenting the first eight tones in an overtone series based on the note C as well as giving the names of the notes and their corresponding frequencies.

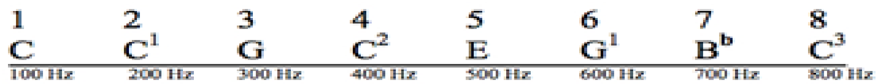

Figure 2. First eight tones in the overtone series and names with frequencies

Figure 3 is similar to Figure 2 but includes additional material. It introduces both the musical intervals as they naturally occur in the overtone series (Martineau, 2011, 247-250), plus the mathematical ratios for the first eight notes in the overtone series. The note C, as the fundamental pitch, is the first note in the series. The second pitch, C^1 (vibrating at 200Hz), lies one octave higher. Therefore, the octave has a ratio of 2:1. Pitches 2 and 3 form the interval of a perfect fifth. Therefore, the perfect fifth has a ratio of 3:2. Pitches 3 and 4 form the musical interval a perfect fourth. Its ratio is 4:3. And intervals 4 and 5 form a major third, with a ratio 5:4. This pattern is consistent throughout the entire overtone series. The musical intervals and their corresponding ratios continue in this pattern.

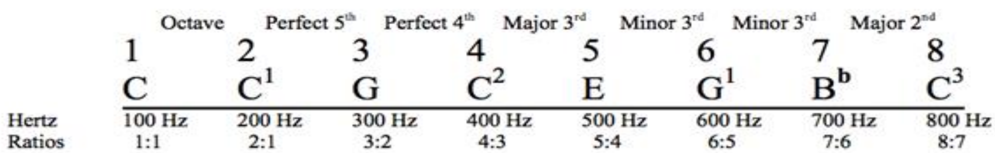

Figure 3. Musical intervals and their corresponding mathematical ratios

As illustrated earlier, the first and second notes in the overtone series constitute an octave. The frequency of the second note is twice that of the first note, the fundamental pitch. The frequency of the fourth pitch in the overtone series is twice that of the second pitch. The second pitch vibrates at 200 Hz and the fourth pitch vibrates at 400 Hz. Therefore, their ratio is 2:1, which constitutes another octave. All musical octaves have a ratio of 2:1. The third note, the note G, vibrates at 300 Hz. The sixth note is also a G (G^1) that lies one octave higher, vibrating at 600 Hz. The fourth note in the series is C^2, vibrating at a frequency of 400 Hz. The eighth note in the series, the note C^3, lies one octave above the fourth note in the series and vibrates at 800 Hz. In short, each octave has a 2:1 ratio. A series of octaves represents a series of 2:1 ratios. A series of octaves can also be described as a geometric sequence since the frequencies of the pitches where the octaves occur are always doubled, i.e., they are multiplied by a factor of 2 (Pierce, 2018b).

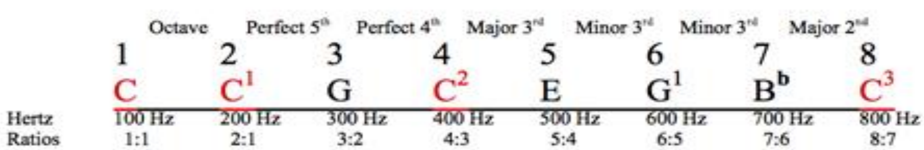

Figure 4. Location of the octaves within the first eight notes

Figure 5 presents the first 16 pitches in an overtone series based on the note C. Notice that each note is numbered in the sequential order in which they naturally occur. Above the number sequence is the name of each musical interval that corresponds to the tonal relationships of the pitches in the sequence. The musical intervals, notated at the top of the illustration, begin with an octave, which is the largest interval. Each subsequent interval gets progressively smaller.

		Octave	Perfect 5th	Perfect 4th	Major 3rd	Minor 3rd	Minor 3rd	Major 2nd	Major 2nd	Major 2nd	Major 2nd	Minor 2nd	Minor 2nd	Minor 2nd	Minor 2nd	Minor 2nd
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	C	**C^1**	G	**C^2**	E	G^1	B^b	**C^3**	**D**	**E^1**	**F#**	**G^2**	**A^b**	**B^{b1}**	**B**	**C^4**
Hertz	100 Hz	200 Hz	300 Hz	400 Hz	500 Hz	600 Hz	700 Hz	800 Hz	900Hz	1000Hz	1100Hz	1200Hz	1300Hz	1400Hz	1500Hz	1600Hz
Ratios	1:1	2:1	3:2	4:3	5:4	6:5	7:6	8:7	9:8	10:9	11:10	12:11	13:12	14:13	15:14	16:15

Figure 5. First 16 pitches in an overtone series

Geometry – Square Root of Two (√2)

Geometry was not a part of my formal education, either in elementary, high school or college. As an adult, a good friend of mine explained what was generally perceived to be a prevailing viewpoint of geometry:

> These are the basic elements of geometry: point, line, plane. It begins with a point, which serves as the fundamental element in geometry. An infinite number of points taken together produce a line. As this process continues, it produces a plane or surface, etc…

It was only later when reading the book *Sacred Geometry* by mathematician and philosopher Robert Lawlor (1989) that the above-mentioned understanding of geometry was supplanted by one that is much more inclusive and intuitive. The fact that he actually expressed *ideas* through geometry made his book, and even the subject of geometry in general, appealing and very interesting. Lawlor had been influenced by the noted philosopher and Egyptologist R.A. Schwaller de Lubicz. It was de Lubicz's understanding of geometry that contrasted with the view expressed above. He stated: "Everything that exists is a volume . . . therefore a point is the apex of a volume, a line is the edge of a volume, and a surface is the face of the volume" (Schwaller De Lubicz R, 1986, 21-22).

Consequently, further investigation led to a striking relationship between the overtone series, which is discussed above, and a basic geometric sequence based on the square root of 2. Just as in the overtone series where a series of octaves were described as a geometric sequence, the geometric pattern based on √2 produces a series of squares where the area of each successive square doubles.

The Process: Diagonalizing the Square

The above examination of the overtone series began with the establishment of a single fundamental *pitch*. Similarly, this examination of geometry will begin with a single fundamental *form*, the square. It starts with a simple line which is of a certain length and has a value of 1, as shown in Figure 6.

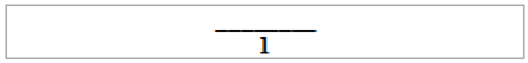

Figure 6. A line – one side of a square, with the value of one

Now to construct a square where all four sides of the square are equal to the original line and also have a value of 1. All of its qualities and attributes have a value of 1, including its area. It can be called a '1' square. Just as with the overtone series where all subsequent pitches in the series of pitches are based on a fundamental pitch, this fundamental square becomes the *fundamental unit* for all subsequent squares, as shown in Figure 6a.

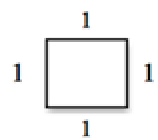

Figure 6a. A fundamental square or a '1' square

The diagonal of this '1' square represents √2, the square root of 2. Diagonalizing this fundamental square creates the possibility of generating new squares based on the properties and proportions of the fundamental square, as shown in Figure 6b.

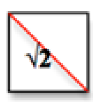

Figure 6b. Diagonal of the '1' square = √2

This diagonal, as the square root of two (√2), now becomes one side of a new square. If we continue to construct a square with sides that are equal to the diagonal of the fundamental square, we will have a new square based on √2, as shown in Figure 6c.

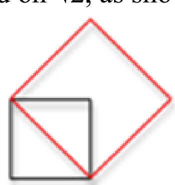

Figure 6c. A new square based on √2

Continuing in this manner, this new square can now be diagonalized, as in Figure 6d.

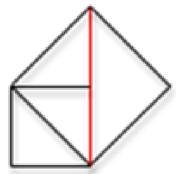

Figure 6d. A diagonalized square

The diagonal of this square in turn becomes the side of a new square. This process creates a progression of squares all based on the √2 of the previous square, as shown in 6e.

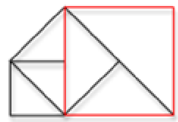

Figure 6e. The side of a new square

By repeating this process, an endless succession of squares can be created that expands by the √2. Figure 6f presents a continuation of this process.

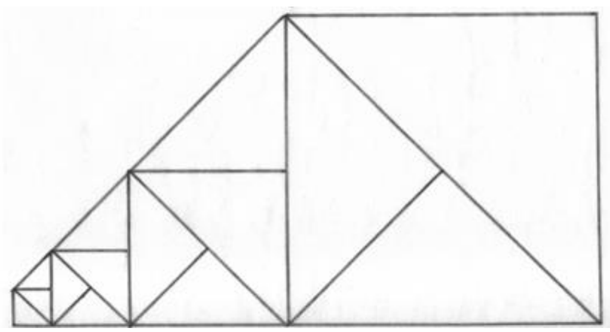

Figure 6f. Potentially endless succession of squares

This progression can continue infinitely. Everything is in direct proportion to the fundamental square, which is derived from the original, fundamental line. The sequence can go in both directions from the infinitesimally small to the infinitely large. The only limit is our ability to reproduce it, considering that it progresses from a point that is too small for us to produce or measure, to a point that exceeds our ability to measure or produce.

The Characteristics

The following examples refer to the process of expansion by doubling, i.e., doubling areas. As before, we begin with a basic fundamental square, where all of the sides are equal to 1, as shown in Figure 7.

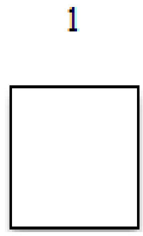

Figure 7. A basic fundamental square

In diagonalizing this square, we have divided it into two equal parts – two isosceles right triangles. These two sides are identical and contain the same area, as shown in Figure 7a.

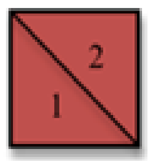

Figure 7a. Diagonalizing the square

Having established the diagonal of this fundamental square, a new square can be constructed whose sides are equal to the diagonal of the fundamental square, as illustrated in Figure 7b.

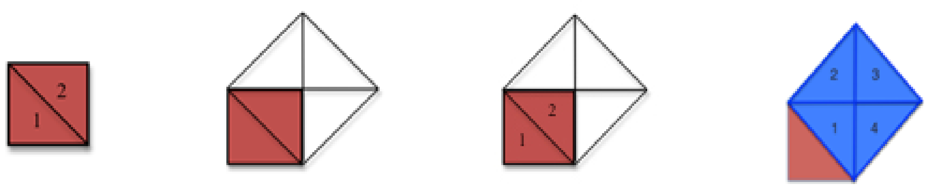

Figure 7b. A new square with four isosceles triangles

Interestingly, this new square contains 4 isosceles triangles. The original fundamental square contained only 2 of these isosceles triangles. Without any complicated mathematics, it is plain to see that this new square contains twice the area of the previous square. In other words, the second square in this sequence has double the area of the previous square. The area of the second square is twice that of the fundamental square. This pattern repeats itself throughout the entire progression, producing areas that expand by doubling with each iteration – each square containing twice the area of the preceding square, and one half of the area of the subsequent square. Stated differently, this is a spatial geometric sequence where the area of each progressive square expands by a 2:1 ratio.

To reiterate the above illustrations with slight variations, and to continue the progression to the next iteration, Figure 7c shows a fundamental square divided into two halves by its diagonal. Each half represents a certain quantifiable area.

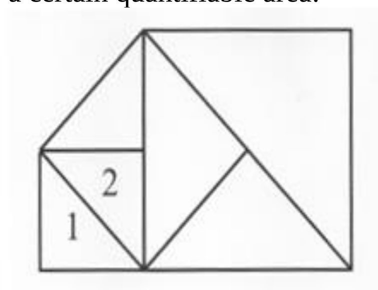

Figure 7c. A fundamental square divided into two halves

The next step in the progression, created when we used the diagonal of the original square as the side of a new square, consist of four of these units of area. Consequently, the area of the second square is twice that of the fundamental square. With each iteration, this pattern repeats itself. Each square in this two-dimensional progression contains twice the area of the previous square and one half the area of the subsequent square.

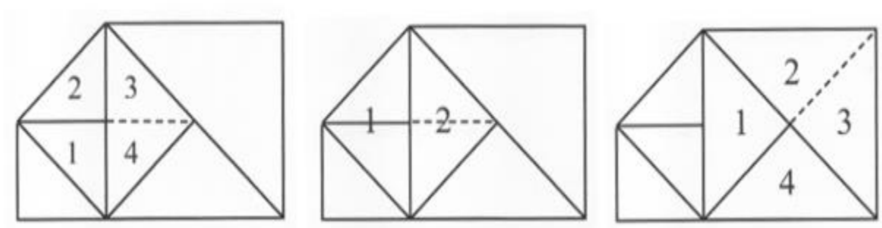

Figure 7d. Doubling areas

The Basics of Space and Time

The overtone series can be viewed as a tonal template, in that it contains all musical pitches. But what is most interesting is the relationship that emerges when the overtone series and the √2 geometric progression are combined.

In Figure 8 we once again have the √2 geometric progression carried out several iterations. As stated previously, each progressive square doubles in area compared to the preceding square and contains one half the area of the subsequent square. In short, it is a progression that perpetually doubles in size. In observing the baseline, along the horizontal line that also serves as a side of each square occurring along this plane (highlighted in red), it can be noted that this baseline doubles in length from the fundamental square to the next square. While the sides of the first square are a certain length, the sides of the second square are twice the length of the first square, and so on. In other words, the length of the sides of each successive square along this baseline is doubling. Since the sides of each individual square are equal, that means that each progressive square is growing at the same exact proportion.

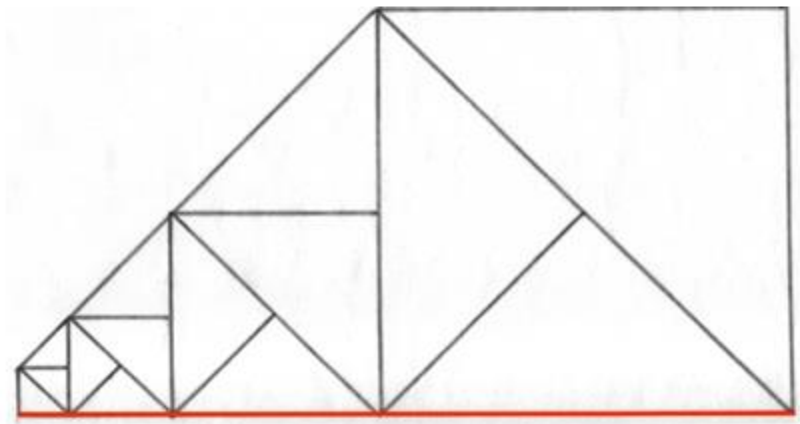

Figure 8. √2 geometric progression that perpetually doubles in size

The next figure, 8a, combines this geometric progression with a series of octaves, i.e., where the octaves would occur if there was an overtone series superimposed. It becomes clearly visible that the points along the baseline where a new square begins (where the length of the baseline side of each square doubles) coincides with the locations in the overtone series where the frequencies double (where octaves occur).

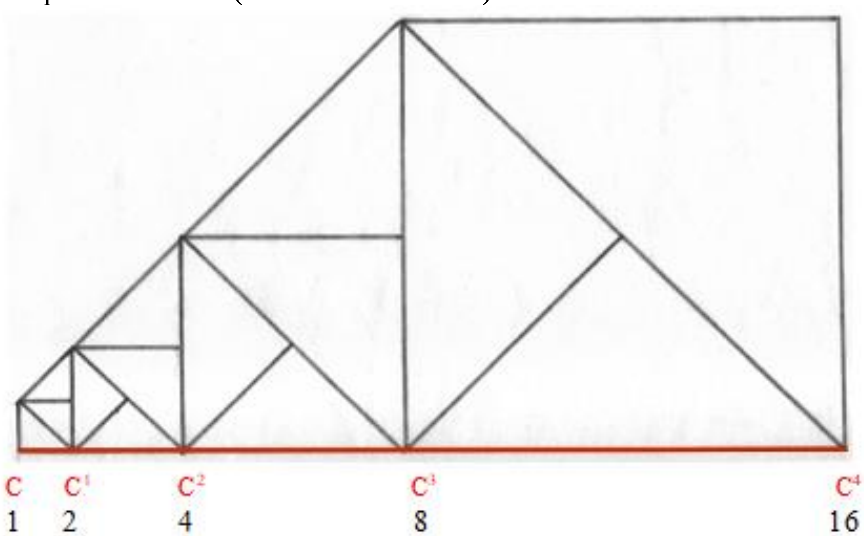

Figure 8a. Octaves combined with the √2 progression

In returning to the previous discussion of the overtone series, Figure 8b illustrates that the number of pitches corresponding to the overtones within each progressive octave continues to increase. For example, within the first octave, from the notes C - C^1, there are no intermediate tones. However, within the second octave, between the notes C^1 - C^2, there is one intermediate tone – the note G. Progressing to the next octave, between the notes C^2 - C^3, there are three intermediate tones: E – G^1 – B^b. The overtone series is clearly expanding, and new tones are emerging to facilitate that expansion.

		Octave	Perfect 5th	Perfect 4th	Major 3rd	Minor 3rd	Minor 3rd	Major 2nd	Major 2nd	Major 2nd	Major 2nd	Minor 2nd	Minor 2nd	Minor 2nd	Minor 2nd	Minor 2nd
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	C	**C^1**	G	**C^2**	E	G^1	B^b	**C^3**	**D**	**E^1**	**F#**	**G^2**	**A^b**	**B^{b1}**	**B**	**C^4**
Hertz	100 Hz	200 Hz	300 Hz	400 Hz	500 Hz	600 Hz	700 Hz	800 Hz	900Hz	1000Hz	1100Hz	1200Hz	1300Hz	1400Hz	1500Hz	1600Hz
Ratios	1:1	2:1	3:2	4:3	5:4	6:5	7:6	8:7	9:8	10:9	11:10	12:11	13:12	14:13	15:14	16:15

Figure 8b. The expanded overtone series

Surprisingly, something new and quite striking emerges when the complete overtone series and the √2 geometric progression are combined. Figure 8c clearly shows a connection between these two phenomena. The rate of expansion of the overtone series directly mirrors the expansion of the squares. With each iteration, some pitches are reiterated (at higher frequencies) while new pitches arise within each octave to serve as the building blocks/ tonal elements of that expansion.

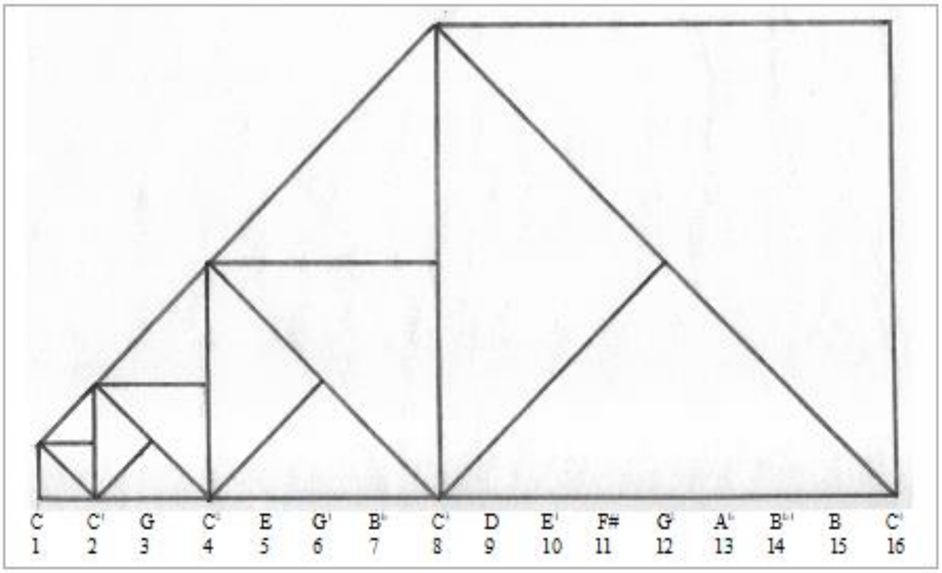

Figure 8c. Overtone series combined with the √2 geometric progression

Previously discussed and illustrated in this paper is the fact that the overtone series and the √2 geometric progression are both based on a single unit. In the overtone series, this single unit is referred to as the *fundamental pitch*. Similarly, the √2 geometric progression begins with a *fundamental square*. In combining the two, the √2 geometric progression and the overtone series, a startling relationship becomes readily recognizable. Just as the overtones in the overtone series are multiples of the frequency of the fundamental pitch, so is the √2 geometric progression based on the area and other special properties of the fundamental square. In other words, the fundamental square of the √2 geometric progression presents itself as the *spatial equivalent* of the fundamental pitch of the overtone series.

In fact, the expansion of the entire overtone series directly mirrors the spatial expansion occurring in the √2 geometric progression. It is possible to surmise that the √2 geometric progression is not only the spatial equivalent of the overtone series, but vice versa: that the overtone series is the temporal equivalent of the spatial expansion represented by the √2 geometric progression.

Even more interesting is exactly where and how the progression depicts the process of doubling. We know from previous examples that in music an octave represents a doubling of a pitch and its corresponding frequency. Each successive octave contained in the overtone series represents a point where the frequency doubles. If we locate the tones in the overtone series where the fundamental pitch is doubled, i.e., where the octaves occur, we will see that these pitches are aligned with the points in the square root of two geometric progression, where the area also doubles. A new pattern and a new relationship between these two very different phenomena thus clearly begins to emerge.

The next series of figures (9, 9a, and 9b) illustrates one of the main and most outstanding features of this relationship between the overtone series and the √2 geometric progression. The first one shows a series of octaves (based on the fundamental pitch C) as they would occur in the overtone series. All of the intermediate tones have been omitted – only the octaves are listed. Below each octave is a number designating the place in the naturally occurring order, where each note/octave would occur in the overtone series. The numbers represent a geometric progression and the pitches represent the corresponding frequencies.

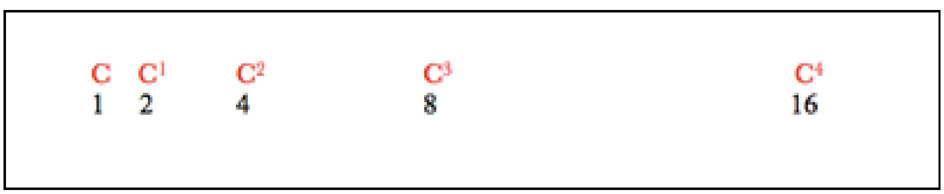

Figure 9. Geometric sequence of tones

Illustrated in Figure 9a is an identical series of pitches and numbers as shown in Figure. 9, but this example is combined with the √2 geometric progression. Here, each octave, where the frequency doubles, corresponds exactly to the points where the area of the squares is doubled.

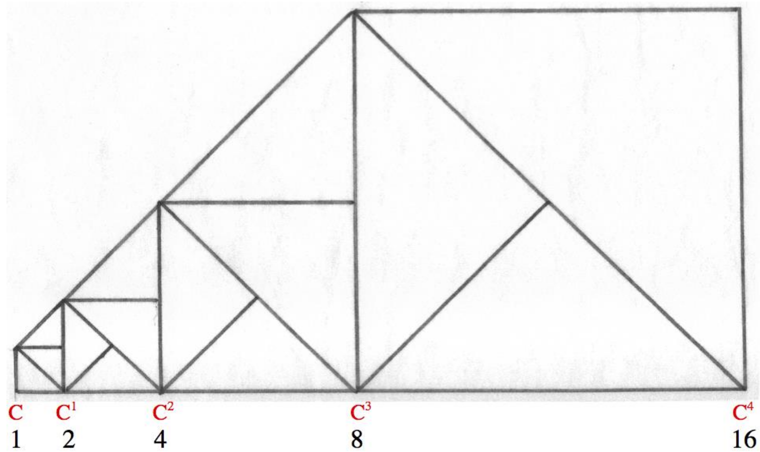

Figure 9a. Geometric sequence of octaves combined with the √2 geometric progression

If the √2 geometric progression is superimposed over the entire overtone series, then a number of remarkable relationships can be observed. Clearly, the points where the squares double correspond exactly to the places where the frequencies double in the overtone series. However, close examination shows that as the overtone series progresses, the number of pitches between each octave increases. Each successive octave contains more and more overtones, i.e., new notes arise in between each progressive octave. The sides of the squares expand in exact accord with the increasing number of overtones. As the overtone series progresses and expands, so does the geometric form. Their growth and expansion match each other with mathematical exactitude and precision.

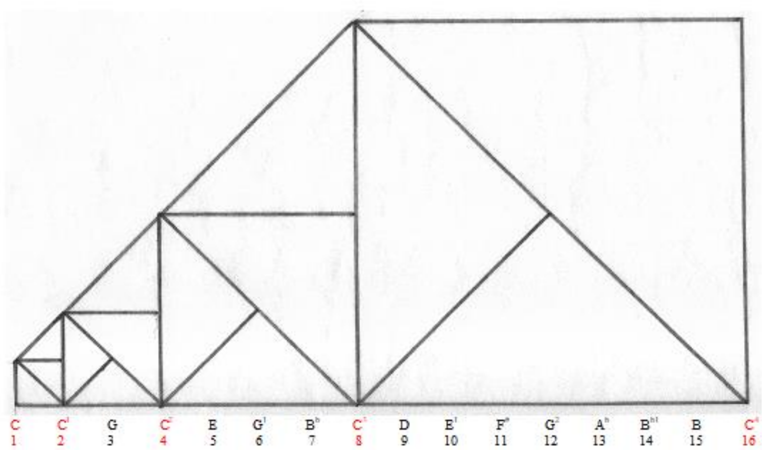

Figure 9b. The overtone series (an arithmetic sequence) and the √2 geometric progression

Figure 9b more fully illustrates the relationship between the overtone series and the √2 geometric progression, a temporal event and a spatial event, respectively. In other words, Figure 9b shows a clear connection between *space and time*. This mathematical relationship evidences a naturally occurring temporal and spatial relationship. The √2 progression can be viewed as the spatial equivalent of the overtone series. The other implications of this relationship will be addressed in future studies.

More about Space and Time

There are a wide variety of musical intervals. In fact, the overtone series contains an infinite number of intervals – from the largest to the smallest. The octave is the first interval and also the largest. After the octave, each musical interval progressively decreases in size (Martineau et al., 2011). There is usually considerable confusion on this issue because Western music is based on equal temperament and under equal temperament we have a limited established set of intervals with the smallest interval being that of a half step. The overtone series is not the result of equal temperament. It is an untempered, naturally occurring sequence of musical tones and intervals – from the largest (the octave) to the most minute.

Another interesting facet of the relationship between the overtone series and the √2 geometric progression is the distinct pattern concerning the order and grouping of the musical intervals. Figure 10 shows the order of the overtones as they naturally occur. This particular overtone series is based on the note C. Each instance in the series of notes where the note C occurs is an octave. You can see that both of the first two notes are the note C, which is *one* octave (C – C^1). The next C that occurs is C^2. However, in the second octave of the series, between C^1 and C^2, there are *two* intervals (C^1 – G and G – C^2) (Perry, 2002). Contained within the next octave, between the notes C^2 and C^3, there are *four* intervals (C^2 – E, E – G^1, G^1 – B^b, B^b – C^3). The next octave, from C^3 to C^4, contains *eight* intervals (C^3 – D, D – E^1, E^1 – $F^\#$, $F^\#$ – G^2, G^2 – A^b, A^b – B^{b1}, B^{b1} – B, B – C^4), as shown in Figure 10.

Just as with the √2 geometric progression, the number of intervals contained in each successive octave doubles. If this overtone series/geometric progression continued, there would be 16 intervals between C^4 and C^5.

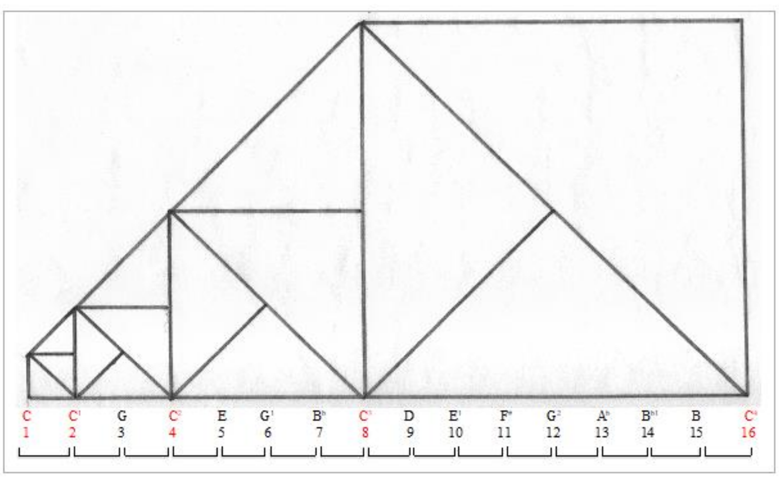

Figure 10. Number of musical intervals doubling within each baseline square

Figure 10a is a slight modification of Figure 10. It shows how the overtone series (an arithmetic sequence) and the √2 geometric sequence combine to form yet another geometric sequence. Each octave contains a number of musical intervals that increase by a factor of 2. As each progressive square along the diagram's baseline doubles in terms of the length of its sides, the overtone series doubles in terms of the number of musical intervals contained within that octave. The first octave has 1 interval, the second octave has 2 intervals. Continuing with the geometric sequence, the third octave contains 4 intervals, and the fourth octave has 8 intervals. If the illustration continued, the next octave would have 16 musical intervals.

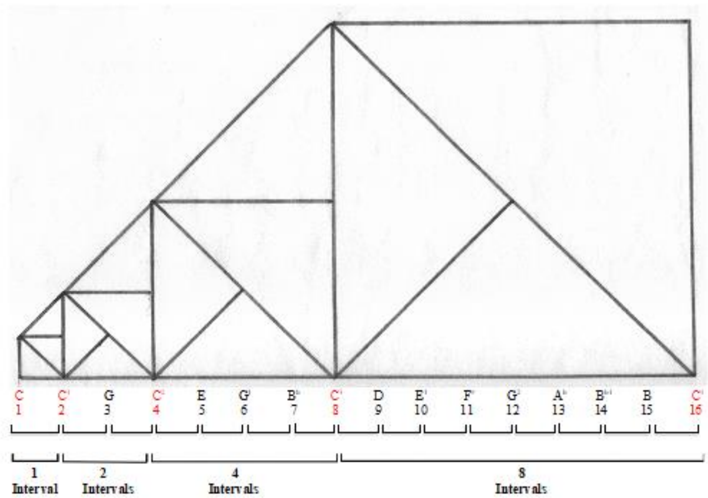

Figure 10a. Doubling intervals (music) and squares (geometry) form a geometric sequence

Astronomy

The implications of the relationship between the overtone series and the √2 geometric progression have been described in this paper as far-reaching. As stated earlier, the Pythagoreans held the following views of arithmetic, music, geometry, and astronomy: *Arithmetic – Number in itself, Music – Number in time, Geometry* -- *Number in space and Astronomy* -- *Number in space and time* (Guthrie, 1987, 34).

The relationship between arithmetic, music, and geometry has been explored in this paper. But what about astronomy? The study of astronomy provides what could prove to be strong evidence of a relationship between the space/time template presented in this paper and what astronomers and astrophysicists have termed orbital resonance. Moons and planets generally orbit a parent body. In the case of moons, the parent body would be a planet. In the case of planets, the parent body would be a star. Orbital resonances occur when gravitational influences affect the orbital relationships of moons or planets, causing them to demonstrate a degree of synchronicity in their orbits. This results in orbital periods occurring in ratios consisting of various size integers. Although any objects orbiting larger objects could possibly have orbital relationships matching any mathematical ratios, astronomers are discovering an increasing number of occurrences of lunar and planetary orbits demonstrating synchronicities that correspond to the small number integers presented here in the overtone series, such as 2:1, 3:2, 4:3, etc. These are the simplest and most noticeable instances of orbital resonances (Aschwanden, 2018; Malhotra, Holman and Ito, 2001; Malhotra, 1999).

Several examples of lunar and planetary orbital resonances can be found in our own solar system. For example, three of Jupiter's Galilean moons, Ganymede, Europa, and Io, orbit Jupiter in a 1:2:4 ratio (Impey, 2018; Lissauer, 2006). For every single orbit completed by Ganymede, Europa completes two. And for every orbit completed by Europa, Io completes two. This series of lunar resonances corresponds to a series of musical octaves. The planet Saturn's moons, Titan and Hyperion, have a 4:3 orbital resonance, meaning that for every four orbits Titan completes around the planet Saturn, Hyperion completes three orbits (Greenberg, Counselman and Shapiro, 1972). This orbital resonance corresponds to a perfect fourth in music. Another two of Saturn's moons, Mimas and Tethys, demonstrate a 2:1 orbital resonance. Mimas completes two orbits around Saturn for every single orbit completed by Tethys. Likewise, the Saturnian moons Enceladus and Dione also exhibit a 2:1 resonance, with Enceladus completing two orbits for every single orbit completed by Dione (University of Toronto, 2017). Each of these two orbital resonances corresponds to a perfect octave in music.

The article *Astrophysicists Convert Moons and Rings of Saturn Into Music* highlights the research of Matt Russo, astrophysicist and postdoctoral researcher at the Canadian Institute for Theoretical Astrophysics (CITA) at the University of Toronto. Russo, along with astrophysicist and postdoctoral researcher Dan Tamayo, and Toronto musician Andrew Santaguida, provide a straightforward description of orbital resonance, stating that orbital resonances "… occur when two objects execute different numbers of complete orbits in the same time, so that they keep returning to their initial configuration" (University of Toronto, 2017). They also credit gravity as the force acting between them that binds them in a repeating pattern.

Russo, Tamayo, and Santaguida use the resonances of Saturn to make connections based on data collected by the Cassini spacecraft to show various relationships between Saturn's moons and rings in comparison with musical tones and intervals. The team actually converted the orbits of various bodies orbiting Saturn into musical notes. By accelerating the speeds of the orbits of Saturn's six large inner moons by 27 octaves, they were able to create pitches within the range of human hearing. In doing so they were able to measure and recognize orbital resonances demonstrated by these moons that correspond to the small number integers mentioned here in the overtone series. These include the following:

2:1 resonance ratio (musical octave)
3:2 resonance ratio (perfect 5th)
4:3 resonance ratio (perfect 4th)
5:4 resonance ratio (major 3rd)
etc…

These examples refer to lunar orbital resonances in our solar system. There are also planetary orbital resonances. For example, the 3:2 resonance between Neptune and Pluto corresponds to a perfect fifth in music. In his article titled, *Self-Organizing Systems in Planetary Physics: Harmonic Resonances of Planet and Moon Orbits*, Aschwanden (2018), an astrophysicist, states that solar systems and planetary systems where the planets and moons are arranged in regularly spaced resonance patterns 3:2, 5:3, 2:1, 5:2, 3:1, etc., tend to be more stable and long lived. He gives our own solar system as an example, citing the 3:2 orbital resonance of Neptune and Pluto (musical perfect fifth), the 3:1 resonance of Saturn and Uranus (octave plus a perfect fifth), and the 2:1 resonance of Uranus and Neptune (musical octave) (Aschwanden, 2018).

Orbital resonances have even been observed outside of our solar system (extrasolar). These also demonstrate orbits that correspond to the small number integers presented in the above discussed overtone series, such as 2:1, 3:2, and 4:3. Laughlin (2010), astronomer and astrophysicist at Yale University, writes that high-precision Doppler velocity monitoring has uncovered an extrasolar planetary system where the three outer planets also have a 1:2:4 orbital resonance as they orbit the star Gilese 876. In addition, it has been discovered that the planetary system Kepler-9 contains two planets, Kepler-9b and 9c, that orbit their star in what appears to be a near 2:1 orbital resonance (Laughlin, 2010; Lee, 2017).

Conclusions

This paper illustrates a relationship between a recognized temporal occurrence (the overtone series) and a known spatial measure (√2 geometric progression). The overtone series is commonly known to many disciplines, including physics, engineering and music. Based on a simple mathematical formula, it does not require an advanced knowledge of music or mathematics to understand the principles that govern the overtone series.

The √2 geometric progression is also based on a simple process. It is a common and uncomplicated progression that could easily be described as basic and elementary. Concerning the overtone series and the √2 progression, this paper presents only a general description of how each works. What is significant here is the combining of these two items that reveals a relationship that is not commonly known.

Music education rarely includes an in-depth study of the overtone series. The greater emphasis is on the study of scales, chords, and rhythm. But the quadrivium describes music as number in time and geometry as number in space. The √2 geometric progression shows a process that produces a spatial sequence that doubles in area as it progresses in exact accord with the octaves in the overtone series.

When combined, the overtone series and the √2 progression form a space–time relationship. To say that the √2 geometric progression presents the spatial equivalent of the overtone series is a bold statement. Such claims about space and time are considered to be the intellectual purview of the astronomer, physicist, or mathematician. Coming from a music theorist, this pronouncement may be met by some with skepticism and doubt. But this is not a priori assumption. The mathematics cannot be simply dismissed. The overtone series and √2 progression have such an interrelated relationship and share such fundamental characteristics that they merit a more thorough examination.

Any particular discipline is simply a single lens through which to observe and study the natural world. Physics is a single discipline, a single lens that studies the world and worldly phenomena through a particular lens. While it is a necessary and much appreciated field of endeavor, it is not absolute. It cannot extend beyond what we can physically observe and measure. Mathematics has to do with number and the various aspects of number. But it is also a single lens and does not take into consideration the musical, or more accurately, the total characteristics and behavior of number. Music is also a single lens, and as such, is limited. A truly multidisciplinary study, such as the quadrivium, has the potential to produce a more comprehensive result.

Knowledge of this type exceeds any single discipline. The quadrivium is an amalgamation of four different disciplines. This multidisciplinary perspective allows phenomena to be viewed simultaneously through four different lenses. It clearly was most desirable during the medieval period when the foundation of the Western education system was based on the seven liberal arts.

The overtone series exists in nature. Orbital resonances exist in nature as well, affecting the amount of time that elapses as two or more moons or planets orbit their parent body and return to their original configuration. This is a temporal measurement. The question is whether the amount of space traversed by these orbiting celestial bodies corresponds to the space/time template illustrated in this paper. In other words, do the spatial/temporal relationships presented in this paper correlate to the ratios that exist in orbital resonances?

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